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THE FINITE STRAIN TENSION TORSION TEST OF A THIN-WALLED TUBE OF ELASTIC–PLASTIC MATERIAL

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Abstract—The kinematics and stress analysis of the tension torsion test of a thin-walled tube at finite strain are discussed. The relationships between increments of tension and torque and increments of extension and twist for an elastic–plastic material at finite plastic strain are formulated for the most common constitutive assumption. Some experimental results are discussed in relation to the analysis and further experiments are recommended.

PLASTIC FLOW AT FINITE STRAIN

The Prandtl–Reuss constitutive equations[1], based on the von Mises yield criterion and its associated normal plastic flow rule, are good approximations to the true behavior of many elastic–plastic materials when infinitesimal strains are involved. These equations, phrased in terms of true stress and true rate of strain for an elastically isotropic material are

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p$$

$$\mathbf{D}^e = \frac{1}{2G}\overset{*}{\boldsymbol{\sigma}}{}' + \frac{1}{9\kappa}\mathbf{I}\ \text{trace}\ \dot{\boldsymbol{\sigma}} \tag{1}$$

$$\mathbf{D}^p = \frac{9}{4h}\frac{\boldsymbol{\sigma}'}{\bar{\sigma}^2}\ \text{trace}\ (\boldsymbol{\sigma}'\overset{*}{\boldsymbol{\sigma}})\ \text{if}\ \bar{\sigma} = \sigma_y\ \text{and trace}\ (\boldsymbol{\sigma}\mathbf{D}^p) > 0$$

or

$$\mathbf{D}^p = 0\ \text{if}\ \bar{\sigma} < \sigma_y\ \text{or trace}\ (\boldsymbol{\sigma}\mathbf{D}^p) \le 0$$

where $\mathbf{D}$ is the deformation rate which is the symmetric part of the velocity gradient in the current configuration, the superscripts e and p indicate the elastic and plastic parts respectively, G is the shear modulus, κ is the bulk modulus, $\boldsymbol{\sigma}$ is the true stress with time rate of change indicated by the superposed dot and $\mathbf{I}$ is the identity tensor. The deviator of true stress is $\boldsymbol{\sigma}'$ while $\overset{*}{\boldsymbol{\sigma}}$ is the Jaumann or corotational rate of change of true stress such that

$$\overset{*}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\Omega}\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{\Omega}^T$$

where $\boldsymbol{\Omega} = (\mathbf{L} - \mathbf{L}^T)/2$ is the spin rate and $\mathbf{L}$ is such that $d\mathbf{v} = L\,d\mathbf{x}$ where $\mathbf{v}$ is velocity and $\mathbf{x}$ is the current position. The tensile equivalent stress is $\bar{\sigma} = \sqrt{(\text{trace}((3/2)\boldsymbol{\sigma}'\boldsymbol{\sigma}')}$ and h is the slope of the uniaxial true stress logarithmic plastic strain curve. The tensile equivalent plastic strain rate $\bar{D}^p = \sqrt{(\text{trace}((2/3)\mathbf{D}^p\mathbf{D}^p))}$. There are two branches for the plastic strain rate one of which represents the loading branch for which the plastic work rate must be positive and applies when $\bar{\sigma}$ is at a critical value σ_y that depends, through an isotropic hardening law, on the total equivalent plastic strain. The other branch is the unloading or below yield branch for which $D^p = 0$. A great deal of experimental work has been done to test the various hypotheses that lead to the formulation of eqn (1). A summary of the early work is to be found in Nadai's book[2]. Lode[3] tested iron, copper and nickel hollow cylinders in combinations of tension and internal pressure. Some deviations from Prandtl–Reuss flow were observed even with infinitesimal strains, but Nadai concludes that the experiments validate the use of eqn (1) as governing the plasticity of the materials tested.

The Prandtl–Reuss equations have been stated in (1) in the objective form thought by many to be reasonably accurate even when plastic strains are large. In this connection a usual assumption, which will be adopted here, is that stress levels are always very low compared to elastic moduli. Thus the elastic and plastic deformation rates can be considered to be additive and the more complete analysis of Lee[4] concerning finite elastic as well as plastic strains need not be introduced. Hill's[1] use of the Prandtl–Reuss equations implies that the definition of stress to be used is true stress, although a correction for spin rate must be made to the stress rate in Hill's equations to make the laws objective. This correction was included by Hutchinson[5], Osias and Swedlow[6], and McMeeking and Rice[7]. The rationale for using true stress and true rate of strain or closely related quantities is that the lattice and polycrystalline aggregate are assumed to be basically unaltered by the passage of dislocations except for an increase in dislocation density that induces an isotropic increase in yield stress. However, there is evidence that the Prandtl–Reuss equations are not entirely suitable for large plastic strains. For example, when a metal is rolled so that there is a large reduction in the cross-sectional area, the extreme elongation of the gains may induce both plastic and elastic anisotropy. Further experimental work is required to determine the limits on the use of the Prandtl–Reuss equations when plastic strains are large

Ideally the experiments should be of the type performed by Taylor and Quinney[8] in which thin-walled tubes were loaded in tension into the plastic range then partially unloaded and twisted until some further plastic flow occurred. The torque twist or torque extension diagrams were extrapolated back to zero twist or zero extension to establish approximately, but fairly accurately, the torque at which plastic flow recommenced. The ratios of initial twist rate to initial extension rate and initial tube internal volume change rate were determined in a like manner. These experiments had the virtue of determining points on the yield surface for combined tensile and shear stress when the material was in a given state determined by the amount of plastic strain accumulated during the tensile loading. The twist rate and extension rate during the nonproportional loading gave information about the flow law, although Taylor and Quinney[8] ignored the possibility of an elastic increment of strain during plastic flow Deviations from the von Mises yield criterion in terms of true stress and from the associated flow law for an initial increment of true strain were observed, but Hill[1] concludes from Taylor and Quinney's experiments and from other data that the Prandtl–Reuss equations can serve as a "reasonably good first approximation" for elastic–plastic behavior of the materials tested. The levels of strain during the tests of Taylor and Quinney[8] were not large, up to 18.5% during the initial extension, but in another experiment Taylor and Quinney[9] compressed a disc of copper using lubricated plattens to a true strain of almost −4. Later Sherby and Young[10] performed pure torsion tests on the same material and compared the tensile equivalent true stress equivalent true strain curve with the earlier data for strains up to 4. The agreement is not exact, but perhaps is sufficiently close to support the use of the Prandtl–Reuss equations phrased in terms of true stress and strain rate for finite plastic strain of annealed copper. This comparison of pure tension with pure torsion is incomplete in the sense that there is a lack of nonproportional straining and because only the initial yield data from the two tests are obtained for the same state of the material.

Bell[11] and Bell and Khan[12] have done experiments which include nonproportional straining of thin-walled tubes of a variety of alloys. These tests involve reasonably large strains and the deviation of the nonproportional tests from the proportional tests gives some information about the material behavior in a given state upon which different stress rates have been imposed. However, there seem to be no tests by Bell which involve distinct unloading and then reloading to some different stress state at or beyond the current yield surface. Thus there are no tests which seek to determine the shape of the yield surface.

In their experiments on thin-walled annealed copper tubes, Bell and Khan[12] observed that for continual proportional or non-proportional loading a constitutive law of the following form could be used:

$$
\begin{aligned}
\mathrm{d}E &= \frac{4\sqrt{2}}{3\sqrt{3}}\frac{\sigma_0\,\mathrm{d}\Sigma}{\beta^2} \\
\mathrm{d}s &= \frac{4\sqrt{2}}{\sqrt{3}}\frac{S\,\mathrm{d}\Sigma}{\beta^2} \qquad (2) \\
\Sigma^2 &= \sigma_0{}^2 + 3S^2
\end{aligned}
$$

where σ_0 is the nominal tensile stress and S is the torque divided by $2\pi R^2 T$, where R is the undeformed mean radius of the tube and T is the undeformed tube wall thickness. The strain E is the change in length of the tube per unit undeformed length, the angle of twist per unit undeformed length of the tube is s/R, and β is a material constant. The stress S is not the nominal shear stress which would be RS/r where r is the current mean radius of the tube. However, Bell and Khan[12] claim that the data prove that the plasticity law for copper is most conveniently stated in terms of a von Mises criterion based on nominal stress and nominal strain rates. Elsewhere Bell[11] claims that a similar formulation is suitable for other alloys. The condition of the material in the undeformed annealed configuration is believed by Bell to have far reaching consequences for plastic flow of the material, and this leads him to the nominal formulation of the plasticity laws. Of course, the hypothesis is clearly undermined by the improper definition of nominal shear stress. But in addition Σ, even ignoring this error, is not the equivalent nominal stress and dE and ds are not the increments of components of any common nominal strain. At best, Bell's laws have to be regarded as an empirical correlation and as will be seen, bear a close resemblance to true stress-true strain rate laws of the sort commonly assumed to govern plasticity at finite strain. In view of the confusion, the kinematics and stress analysis of the tension torsion test of a thin-walled tube will be discussed in the next sections.

KINEMATICS OF THE TENSION TORSION TEST

Consider a thin-walled tube with wall thickness T formed so that prior to deformation the cross section is circular with radius R to the midpoint of the wall as shown in Fig. 1. The

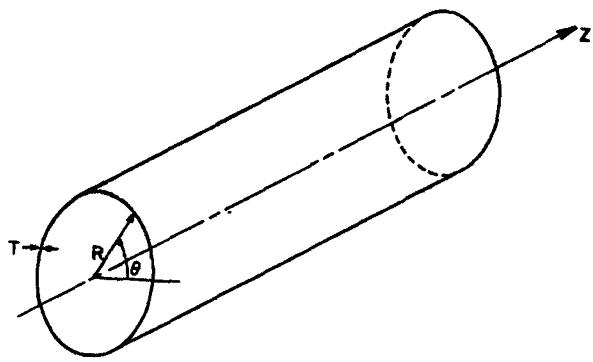

Fig 1

position of material in the tube is given by cylindrical polar coordinates R, θ and Z in the undeformed configuration (see Fig. 1). The tube is assumed to be sufficiently thin so that the deformation in twisting and extension can be regarded as homogeneous through the wall and calculated as the deformation of the midsurface of the wall. During deformation the radius and thickness of the tube are r and t, respectively, and the ring of material originally at Z is rotated by an angle $\omega = sZ/R$ about the Z axis and moved to the axial position $z = (1+E)Z$. In this motion the undeformed elemental length vector $d\mathbf{X}$ becomes the deformed element $d\mathbf{x} = \mathbf{F}\,d\mathbf{X}$. The components of the deformation gradient $\mathbf{F}$ referred to the reference polar coordinate system (R, θ, Z) are given by the product of two matrices

$$[F] = [R_\omega][F_\omega]$$

where

$$[R_\omega] = \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[F_\omega] = \begin{bmatrix} t/T & 0 & 0 \\ 0 & r/R & rs/R \\ 0 & 0 & 1+E \end{bmatrix}. \tag{3}$$

The components of L can be computed from $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$. Expressed in cylindrical polar coordinates (r, ϕ, z) in the current configuration, the components of L are

$$[L] = \begin{bmatrix} \dot{t}/t & -\dot{s}Z/R & 0 \\ \dot{s}Z/R & \dot{r}/r & r\dot{s}/R(1+E) \\ 0 & 0 & \dot{E}/(1+E) \end{bmatrix} \tag{4}$$

For convenience the notation $\epsilon = \dot{E}/(1+E)$ and $\gamma = r\dot{s}/R(1+E)$ is introduced so that $D_{\phi z} = \gamma/2$ and $D_{zz} = \epsilon$.

Bell and Khan claim that their finite strain plasticity theory is phrased in terms of a strain defined so that the principal strains are the principal stretch ratios minus one. This strain can be calculated by first performing a polar decomposition of $\mathbf{F}$ into a symmetric tensor contracted with an orthogonal tensor and then subtracting the identity tensor from the symmetric tensor. When this is done it is found that $\dot{E}$ and $\dot{s}$ are not simultaneously proportional to components of the rate of such a strain in any orthogonal coordinate system.

STRESS ANALYSIS OF THE TENSION TORSION TEST

Let the tension applied to the tube be $2\pi rt\sigma$ and the torque be $2\pi r^2 t\tau$. Assume that the tube wall is sufficiently thin so that the stress state is homogeneous and thus the true or Cauchy stress has the nonzero components

$$\sigma_{zz} = \sigma \quad \text{and} \quad \sigma_{z\phi} = \sigma_{\phi z} = \tau. \tag{5}$$

The nominal or first Piola–Kirchhoff stress is related to the true stress by

$$\boldsymbol{\sigma}_N = (\mathrm{Det}\ \mathbf{F})\mathbf{F}^{-1}\sigma. \tag{6}$$

Thus the components of $\boldsymbol{\sigma}_N$ in (R, θ, Z) undeformed coordinates are given by the product matrix

$$[\sigma_N] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -s\dfrac{R}{r}S & \left(\dfrac{R}{r}\right)^2(1+E)S - \sigma_0 s \\ 0 & \dfrac{R}{r}S & \sigma_0 \end{bmatrix} [R_\omega]^T \tag{7}$$

where $\sigma_0 = \sigma rt/RT$ and $S = r^2t\tau/R^2T$ have been used. The first matrix in the expression for $\boldsymbol{\sigma}_N$ is simply the nominal stress at $Z = 0$.

The tensile equivalent true stress for the tension torsion test is

$$\bar{\sigma} = \sqrt{(\sigma^2 + 3\tau^2)} \tag{8}$$

while the effective nominal stress $\bar{\sigma}_N$, such that $\bar{\sigma}_N^2 = 3/2\ \mathrm{trace}\ (\boldsymbol{\sigma}_N'^T\boldsymbol{\sigma}_N')$, is

$$\bar{\sigma}_N^2 = \sigma_0^2\left(1 + \frac{3}{2}s^2\right) + \sigma_0 S\left(\frac{R}{r}\right)\left[1 - 3\left(\frac{R}{r}\right)(1+E)s\right] + S^2\left(\frac{R}{r}\right)^2\left[1 + \frac{3}{2}s^2 + \frac{3}{2}(1+E)^2\left(\frac{R}{r}\right)^2\right]. \tag{9}$$

Thus Σ is not, in general, proportional to the effective nominal stress even when S^2 is replaced by $(R/r)^2 S^2$ in Σ. However,

$$\Sigma = (rt/RT)\sqrt{(\sigma^2 + 3(r/R)^2\tau^2)} \tag{10}$$

and so, but for the error in defining nominal shear stress, Σ is proportional to $\bar{\sigma}$.

TENSION TORSION TEST AT FINITE PLASTIC STRAIN

It has already been shown that Σ is not the effective nominal stress and is similar to the effective true stress. Bell and Khan[12] did not test for the shape of the yield surface in a given material state after finite plastic strain. In view of the lack of convincing evidence otherwise, the hypothesis that (1) is a reasonably accurate model for plastic flow at finite strain will not be disposed of. The response of the thin-walled tube of material obeying the constitutive law (1) will now be considered.

The nonzero components of $\overset{*}{\sigma}$ are

$$\overset{*}{\sigma}_{zz} = \dot{\sigma} + \tau\gamma \qquad \overset{*}{\sigma}_{\phi\phi} = -\tau\gamma \qquad \overset{*}{\sigma} = \dot{\tau} - \sigma\gamma/2. \tag{11}$$

The elastic–plastic strain rates resulting from these stress rates and the constitutive law (1) are

$$\begin{aligned}
\epsilon &= \left(\sigma + \frac{3\tau^2}{\sigma+2G}\right)\frac{\dot{\bar{\sigma}}}{h\bar{\sigma}} + \left(\frac{1}{3G}+\frac{1}{9\kappa}\right)\dot{\sigma} + \frac{\tau}{\sigma+2G}\frac{\dot{\tau}}{G} \\
\gamma &= \frac{1}{(1+\sigma/2G)}\left(\frac{3\tau\dot{\bar{\sigma}}}{h\bar{\sigma}} + \frac{\dot{\tau}}{G}\right) \\
\frac{\dot{r}}{r} &= -\left(\frac{\sigma}{2}+\frac{3\tau^2}{\sigma+2G}\right)\frac{\dot{\bar{\sigma}}}{h\bar{\sigma}} + \left(\frac{1}{9\kappa}-\frac{1}{6G}\right)\dot{\sigma} - \frac{\tau}{\sigma+2G}\frac{\dot{\tau}}{G} \\
\frac{\dot{t}}{t} &= -\frac{\sigma\dot{\bar{\sigma}}}{2h\bar{\sigma}} + \left(\frac{1}{9\kappa}-\frac{1}{6G}\right)\sigma.
\end{aligned} \tag{12}$$

When elastic strain rates are neglected compared to plastic strain rates and terms of order stress neglected compared to elastic moduli, it is found that

$$\begin{aligned}
\mathrm{d}E &= \sigma_0(1+E)\frac{RT}{rt}\left(\frac{\mathrm{d}\bar{\sigma}}{h\bar{\sigma}}\right) \\
\mathrm{d}s &= 3\left(\frac{R}{r}\right)^2 S(1+E)\frac{RT}{rt}\left(\frac{\mathrm{d}\bar{\sigma}}{h\bar{\sigma}}\right).
\end{aligned} \tag{13}$$

This means that

$$\mathrm{d}s/\mathrm{d}E = (R/r)^2 3S/\sigma_0 \tag{14}$$

while the law of Bell and Khan[12] gives

$$\mathrm{d}s/\mathrm{d}E = 3S/\sigma_0. \tag{15}$$

Thus the empirical fit of Bell and Khan in eqn (2) is not predicted by the Prandtl–Reuss equation (1). Note that Bell's data seem to have been obtained for stretch ratios up to 1.3. When volume change is neglected $(R/r^2) = 1+E$ and so the factor $(R/r)^2$ in (14) could be as high as 1.3. The discrepancy between (14) and (15), if real, may make eqn (1) too inaccurate for use in describing plastic flow in copper. On the other hand, it may be an acceptable approximation, and it remains to be seen whether Bell and Khan's empirical fit is good for much larger or much smaller stretch ratios of order 2 or more or 0.5 or less. As emphasized previously, it would be preferable for the tests to determine this and other features of the constitutive law to be based on the method of Taylor and Quinney[8]. The specimen should be stressed into the large plastic strain range, unloaded and then reloaded beyond the yield surface in a different stress state. In this way a distinct test for the yield surface shape can be made. Also, a comparison of the distinct values of $\mathrm{d}s/\mathrm{d}E$ for the proportional test and for the nonproportional test can be made which would perhaps be more useful than a test involving a gradual change of $\mathrm{d}s/\mathrm{d}E$ during nonproportional but continuous plastic loading. During nonproportional loading from the stress point at which previous plastic loading took place, the ratio $\mathrm{d}s/\mathrm{d}E$ could be greatly influenced by the details of the local shape of the yield surface. There could be a vertex on the yield surface induced by the proportional loading and perhaps this would influence ds/dE greatly even if the vertex was not a very prominent feature of a yield surface basically von Mises in shape. That vertices on the yield surface are a reasonable possibility has been shown by Hutchinson[13] in his analysis of plastic flow in a polycrystalline aggregate.

In closing, it should be re-emphasized that the Prandtl–Reuss equation (1) is only expected to be an approximation to the true flow behavior of the elastic–plastic materials. Experiments performed as carefully as those of Bell and coworkers are required to determine not just whether the eqn (1) is reasonably accurate in some circumstances but also when the equations are inaccurate. However, proper analysis of the data is necessary. Analysis of the results of Bell and Khan[12] reveals that the true stress-true strain rate Prandtl–Reuss equations seem to be justified to within a degree of approximation.

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REFERENCES

1. R. Hill, *The Mathematical Theory of Plasticity.* Oxford University Press (1950).
2. A. Nadai, *Theory of Flow and Fracture of Solids*, 2nd Edn, Vol. 1, Chap 17 McGraw-Hill, New York (1950).
3. W. Lode, Versuche ueber den Einfluss der mittleren Hauptspannung auf die Fliessgrenze. *Ztschr F Angew Math. und Mech.* **5**, 142–144 (1925) and papers cited by Nadai [2]
4. E. H. Lee, Elastic–Plastic deformations at finite strain. *J. Appl. Mech.* **36**, 1 (1969).
5. J. W Hutchinson, Finite strain analysis of elastic–plastic solids and structures. *Numerical Solution of Nonlinear Structural Problems* (Edited by R. F. Hartung) p 17. ASME, New York (1973)
6. J. R. Osias and J. L Swedlow, Finite elastoplastic deformation, I, theory and numerical examples *Int. J. Solids Structures* **10**, 321 (1974).
7. R. M. McMeeking and J. R. Rice, Finite-element formulations for problems of large elastic–plastic deformation. *Int. J. Solids Structures* **11**, 601 (1975).
8. G. I. Taylor and H. Quinney, The plastic distortion of metals. *Phil. Trans. Royal Soc. A* **230**, 323 (1931).
9. G. I Taylor and H. Quinney, The latent energy remaining in a metal after cold working. *Proc. Royal Soc. A* **143**, 307 (1934).
10. O. D. Sherby and C. M. Young, *Rate Processes in Plastic Flow* (Edited by J. C. M. Li and A. K Mukherjee). American Society for Metals, Metals Park, Ohio (1975)
11. J. F Bell. A physical basis for continuum theories of finite strain plasticity, I & II *Archive for Rational Mechanics and Analysis*: Part I **70**, 319 (1979), Part II **75**, 103 (1981).
12. J. F. Bell A. S. Khan, Finite plastic strain annealed copper during non-proportional loading. *Int. J. Solids Structures* **16**, 683 (1980).
13. J. W Hutchinson, Elastic–plastic behavior of polycrystalline metals and composites. *Proc Royal Soc. A* **319**, 247 (1970).